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ATTRACTOR OF AN ITERATED FUNCTION SYSTEM HAVING CYCLIC ĆIRIĆ-REICH-RUS CONTRACTIONS IN QUASI-PARTIAL *B*-METRIC SPACE

M.Priya*

Department of Mathematics, The Oxford College of Science, Arts, Commerce and Management, Bangalore, Karnataka, India

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Abstract: This endeavor exploits the common fixed point theorem for cyclic Ćirić-Reich-Rus contraction mappings in quasi-partial *b*-metric space to explore an iterated function system for the above said contractions in the quasi-partial *b*-metric space. Expending the standard procedure of constructing deterministic fractals (Attractors) from an iterated function system is considered. Also a compact invariant subset of an iterated function system consisting a particular operator defined on the hyperspace of all non-empty compact subsets of a compact subspace, which is a subset of a complete quasi-partial *b*-metric space is established.

Introduction

The groundwork for the early development of fractal geometry was provided by the philosopher and mathematician Leibiniz in the 19^{th} century. The graph of a function which should be considered as a fractal was presented by Weierstrass nearly two hundred years elapsed before. That is, it can be represented a function which is everywhere continuous but nowhere differentiable. The French-

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TOJSR. 2025, 4 (1)

American mathematician Benoit Mandelbrot has gained a victory in combining hundreds of years of mathematical research by pointing the word "fractal", with the aid of his predecessors. Most of the real life objects are fractals and they have self-similarity in nature. This is not only a property of a fractal but also used to define them.

Iterated function system is a method of generating fractals using self-similarity. A rich source of fractals is iterated function system. iterated function system is the base for fractal image compression techniques. One of the most common ways of generating fractals is as the fixed attractor set of an iterated function system using Hutchinson-Barnsley theory (HB theory, in short). Various kinds of iterated function system would be found in [12, 13, 14]. In 2013, Uthayakumar and Arockia Prabakar [15] introduced the R iterated function system and governed the HB theory to create a new fractal set as its unique fixed point by exploiting Reich contractions in a complete *b*-metric space.

In 1905, Frechet started the exploration of metric spaces and its properties[2]. After his study of metric spaces, several types of metric spaces were introduced and algebraic and analytical properties of such metric spaces were examined by various mathematicians. One of the most famous metric spaces in the metric space theory is quasi-metric space and partial metric space. A common idea of standard metric spaces is known as quasi-metric spaces and it was revealed by Wilson[1]. A relaxation in the symmetric property of a usual metric leads to a generalization of this concept called a quasi metric. Further, it is more useful for solving different real-life phenomena. That is, researchers are trying to model these issues into a problem having a unique solution such as $\phi(x) = x$. And some of the exemplifications of such study can be found in [25, 26, 27].

Replacing the condition $d(x, x) = 0$ with the condition $d(x, x) \leq d(x, y) \forall x, y \in X$ in the definition of standard metric space was considered by Matthews [3] in his research. Theoretical applications of quasi-metric spaces can be found in computer science. The idea of *b*-metric space was derived by Bakhtin [5] and it was extended by Czerwik [4]. A generalized version of metric space is encountered with the notion of *b*-metrics. Moreover, unlike the normal metric *b*-metric need not be continuous due to the modification of the triangle inequality [22]. However, there are several fixed point theorems are performed in the *b*-metric spaces both for the point-valued contractions and set-valued contractions [23, 24]. By utilizing the concept of fixed point theorem, generally known as the Banach contraction principle, the existence of a solution to an integral equation was established in 1922. The development of fixed point theory in ordinary metric spaces gained an attention of the mathematicians in the end decades of 20th century.

The notion of quasi-partial metric spaces, as a further generalization for the quasi metric spaces and partial metric spaces, was initiated by Karapinar *et al.* [6] and they discussed the existence of fixed points of self-mappings on quasi-partial metric spaces. An intensive study on the collection of quasi-partial *b*-metric spaces and on the fixed points

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TOJSR. 2025, 4 (1)

of some self-mappings introduced by Karapinar that have a close relationship with T-orbitally lower semi-continuous functions was made by Gupta and Gautam [7]. Ćirić [8] gave the definition of quasi-contraction mappings and stated some fixed point theorems. In 2003, the cyclic contraction mapping was introduced by Kirk et al [9]. The existence and uniqueness of common fixed points for cyclic Ćirić-Reich-Rus mapping [10] was discussed by Vishnu Narayan Mishra *et al.* [11].

The current research develops the HB theory using cyclic Ćirić-Reich-Rus contractions as the iterated function system on a complete quasi-partial b-metric space. The present work is employed as follows: Section 2 presents the basics and preliminaries about quasi-partial b-metric space and cyclic Ćirić-Reich-Rus contractions. Also we can find the classical HB theory in the same section. In section 3, some basic lemmas and results are prepared to taste the HB theory. Section 4 serves the notion of attractor of the undertaken iterated function system and it also gives a new methodology for constructing an attractor of a self-mapping defined on the hyperspace of a complete quasi-partial b-metric space. Finally, the summary of this paper is given in section 5.

Basics and Preliminaries

This section establishes the fundamental definitions and results of quasi-partial b-metric spaces, cyclic Ćirić-Reich-Rus contraction mappings and some related fixed point theorems. For developing the Hutchinson-Barnsley theory using the cyclic Ćirić-Reich-Rus contractions, we need to show the concept of classical HB theory in the same section.

Common fixed point theorem for cyclic Ćirić-Reich-Rus contraction mappings in quasi-partial b-metric space

Definition 0.0.1. *[7] A quasi-partial b-metric on a non-empty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$,*

(1) $qp_b(x, x) = qp_b(x, y) = qp_b(y, y)$ implies $x = y$.

(2) $qp_b(x, x) \leq qp_b(x, y)$,

(3) $qp_b(x, x) \leq qp_b(y, x)$,

(4) $qp_b(x, y) + qp_b(z, z) \leq s[qp_b(x, z) + qp_b(z, y)]$.

A quasi-partial b-metric space is a pair (X, qp_b), where X is a empty space and qp_b is quasi-partial b-metric on X. The number s is called coe cient of (X, qp_b).

Definition 0.0.2. *[7] Let (X, qp_b) be a quasi-partial b-metric space. Then the following holds:*

(i) A sequence $\{x_n\} \subset X$ is said to convergent to $x \in X$, if

$$qp_b(x, x) = \lim_{n\to\infty} qp_b(x_n, x) = \lim_{n\to\infty} qp_b(x, x_n).$$

ISSN: 2581 - 9380
TOJSR. 2025, 4 (1)

(ii) *A sequence $\{x_n\} \subset X$ is called a Cauchy sequence, if*

$$\lim_{n,m\to\infty} qp_b(x_n, x_m) = \lim_{m,n\to\infty} qp_b(x_m, x_n) \text{ exists and finite.}$$

(iii) *The quasi-partial b-metric space (X, qp_b) is said to be complete, if every sequence $\{x_n\} \subset X$ converges with respect to τ_{qp_b} to a point $x \in X$ such that*

$$qp_b(x, x) = \lim_{n,m\to\infty} qp_b(x_n, x_m) = \lim_{m,n\to\infty} qp_b(x_m, x_n).$$

Lemma 0.0.1. *[7] Let (X, qp_b) be a quasi-partial b-metric space and $\{x_n\}^{\infty}_{n=1}$ be a sequence in X. If $x_n \to x, x_n \to y$ and $qp_b(x, x) = qp_b(y, y) = 0$, then $x = y$.*

Definition 0.0.3. *[11] Let (X, qp_b) be a quasi-partial b-metric space. A sequence $\{x_n\} \subset X$ is said to convergent to $x \in X$, if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$ such that $x_n \in B(x, \epsilon)$, for all $n \geq K(\epsilon)$.*

Remark 0.0.1. *It is not hard to check that the Definitions 0.0.2 and 0.0.3 are equivalent and both give the notion of convergence of a sequence in the quasi-partial b-metric space.*

Lemma 0.0.2. *[11] Let X be a quasi-partial b-metric space and $A \subset X$, $f : A \to X$ is said to be continuous at a point $x_0 \in A$ if and only if for every sequence $\{x_n\}$ in A that converges to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.*

Definition 0.0.4. *[16] Let S and T be self maps on a non-empty set X. If there exists $x \in X$ such that $Sx = Tx$, then x is called a coincidence point of S and T, while $y = Sx = Tx$ is called a point of coincidence (or coincidence value) of S and T. If $Sx = Tx = x$, then x is called a common fixed point of S and T.*

Definition 0.0.5. *[17] Let S and T be self maps on a non-empty set X. The pair of mappings S and T is said to be weakly compatible, if they commute at their coincidence points.*

Definition 0.0.6. *[11] Let (X, qp_b) be a quasi-partial b-metric space. Let A and T be continuous self maps on X that commutes and $T(X) \subset A(X)$, further, let A and T satisfy the following: for any $x, y \in X$,*

$$qp_b(Tx, Ty) \leq \alpha qp_b(Ax, Ay) + \beta qp_b(Ax, Tx) + \beta qp_b(Ay, Ty),$$

where $\alpha, \beta \in (0, 1)$ such that for $s \geq 1$ and $s(\alpha + 2\beta) < 1$. Then the map T is called the cyclic Ćirić-Reich-Rus contraction mapping on the quasi-partial b-metric space (X, qp_b). Here α, β are called the contractivity ratios of the contraction mapping.

Theorem 0.0.1. *[11] Let (X, qp_b) be a complete quasi-partial b-metric space. Let A and T be continuous self maps on X that commutes and $T(X) \subset A(X)$, further, let A and T satisfy the following: for any $x, y \in X$,*

$$qp_b(Tx, Ty) \leq \alpha qp_b(Ax, Ay) + \beta qp_b(Ax, Tx) + \beta qp_b(Ay, Ty),$$

where $\alpha, \beta \in (0, 1)$ such that for $s \geq 1$ and $s(\alpha + 2\beta) < 1$, A and T have a unique fixed point.

TOJSR The Oxford Journal of Science and Research

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TOJSR. 2025, 4 (1)

Classical HB theory

The contention of fractal geometry is studied always in an ideal space known as $(\mathbb{H}(X), h)$, where $\mathbb{H}(X)$ denotes the space whose points are the compact subsets of X, other than the empty set and h is a suitable metric on the set $\mathbb{H}(X)$. Now we shall see the development of such space.

Definition 0.0.7. *[18] Let (X, d) be a metric space with distance function d and T be a mapping from X into itself. Then T is called a contraction mapping if there is a constant $0 \leq k < 1$ such that*

$$d(Tx, Ty) \leq kd(x, y)$$

$\forall x, y \in X$. The constant k is called contractivity factor for T.

Theorem 0.0.2. *[18] Let $T : X \to X$ be a contraction mapping, with contractivity factor k, on a complete metric space (X, d). Then T possesses exactly unique fixed point $x \in X$.*

Definition 0.0.8. *[20] Let (X, d) be a metric space and $\mathbb{H}(X)$ be the set of all non-empty compact subsets of X. Define $d(a, B) = inf\{d(a, b) : b \in B\}$ and $d(A, B) = sup\{d(a, B) : a \in A\}$. Then the Hausdorff metric or Hausdorff distance $h(A, B)$ is a function $h : \mathbb{H}(X) \times \mathbb{H}(X) \to \mathbb{R}$ defined by*

$$h(A, B) = max\{d(A, B), d(B, A)\}.$$

Then h is a metric on the hyperspace of compact sets $\mathbb{H}(X)$ and hence $(\mathbb{H}(X), h)$ is called a Hausdorff metric space.

Theorem 0.0.3. *[19, 20] If (X, d) is a complete metric space then $(\mathbb{H}(X), h)$ is also a complete metric space.*

Definition 0.0.9. *[19, 20] Let (X, d) be a metric space and $T_n : X \to X, n = 1, 2, \ldots, N_0 (N_0 \in \mathbb{N})$ be N_0 contraction mappings with the contractivity ratios p_n, $n = 1, 2, \ldots, N_0$. The system $\{X; T_n, 1, 2, \ldots, N_0\}$ is called an Iterated Function System or Hyperbolic Iterated Function System with the ratio $p = \max_{1 \leq n \leq N_0} p_n$. Then the Hutchinson-Barnsley operator (HB operator) of the iterated function system is a function $\hat{T} : \mathbb{H}(X) \to \mathbb{H}(X)$ defined by $\hat{T}(B) = \bigcup_{n=1}^{N_0} T_n(B)$, for all $B \in \mathbb{H}(X)$.*

Theorem 0.0.4. *[19, 20] Let (X, d) be a metric space. Let $\{X; T_n, n = 1, 2, \ldots, N_0\}$ be an iterated function system. Then, the HB operator $\hat{T}$ is a contraction mapping on $(\mathbb{H}(X), h)$.*

Theorem 0.0.5. *[19, 20] Let (X, d) be a metric space. Let $\{X; T_n, n = 1, 2, \ldots, N_0\}$ be an iterated function system. Then there exists only one compact invariant set $A_\infty \in \mathbb{H}(X)$ of the HB operator $\hat{T}$ (or) equivalently $\hat{T}$ has a unique fixed point $A_\infty \in \mathbb{H}(X)$.*

The fixed point $A_\infty \in \mathbb{H}(X)$ of the HB operator $\hat{T}$ described in Theorem 0.0.5 is called an attractor (Fractal) of the iterated function system. So $A_\infty \in \mathbb{H}(X)$ is called a fractal generated by iterated function system of classical Banach contraction [19, 20].

Definition 0.0.10. *[20] Let (X, d) be a metric space and let $C \in \mathbb{H}(X)$. Define a transformation $T_0 : \mathbb{H}(X) \to \mathbb{H}(X)$ by $T_0(B) = C$ for all $B \in \mathbb{H}(X)$. Then T_0 is called a condensation transformation and C is called the associated condensation set.*

TOJSR The Oxford Journal of Science and Research

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TOJSR. 2025, 4 (1)

Note 0.0.1. *[20] A condensation transformation $T_0 : \mathrm{H}(X) \rightarrow \mathrm{H}(X)$ is a contraction mapping on the metric space $(\mathrm{H}(X), h(d))$, with contractivity ratio equal to zero, and it possesses a unique fixed point, namely condensation set.*

Definition 0.0.11. *[20] Let $\{X; T_n, n = 0, 1, 2, \ldots, N_0\}$ be a hyperbolic iterated function system with contractivity ratio $0 \leq p < 1$. Let $T_0 : \mathrm{H}(X) \rightarrow \mathrm{H}(X)$ be a condensation transformation. Then $\{X; T_n, n = 0, 1, 2, \ldots, N_0\}$ is called a hyperbolic iterated function system with condensation with contractivity ratio p.*

Theorem 0.0.4 and Theorem 0.0.5 can be modified to cover the case of an iterated function system with condensation.

Theorem 0.0.6. *[20] Let $\{X; T_n, n = 0, 1, 2, \ldots, N_0\}$ be a hyperbolic iterated function system with condensation, with contractivity ratio p. Then the transformation $\hat{T} : \mathrm{H}(X) \rightarrow \mathrm{H}(X)$ defined by $\hat{T}(B) = \bigcup_{n=0}^{N_0} T_n(B)$, for all $B \in \mathrm{H}(X)$ is a contraction mapping on the complete metric space $(\mathrm{H}(X), h(d))$ with contractivity ratio p. That is*

$$h(T(B), T(C)) \leq p.h(B, C) \forall B, C \in \mathrm{H}(X).$$

Its unique fixed point, $A \in \mathrm{H}(X)$, obeys

$$A = T(A) = \bigcup_{n=0}^{N_0} T_n(A)$$

and is given by $A = \lim_{n\to\infty} T_n(B)$ for any $B \in \mathrm{H}(X)$.

Preparations

Some basic lemmas and theorems are presented to construct the HB theory having cyclic Ćirić-Reich-Rus contraction mapping as an iterated function system. First we define the Hausdorff distance of two sets in the quasi-partial b-metric space.

Definition 0.0.12. *Let (X, qp_b) be a quasi-partial b-metric space. Suppose $\mathrm{H}(X)$ is the set of all compact subsets of the quasi-partial b-metric space X. Now we define $d_{qp_b}(a, B) = \inf\{qp_b(a, b) : b \in B\}$ and $d_{qp_b}(A, B) = \sup\{qp_b(a, B) : a \in A\}$. Then the Hausdorff distance of two subsets A, B of the quasi-partial b-metric space (X, qp_b) is a mapping $h_{qp_b} : \mathrm{H}(X) \rightarrow \mathrm{H}(X)$ given by*

$$h_{qp_b}(A, B) = \max\{d_{qp_b}(A, B), d_{qp_b}(B, A)\}$$

This function h_{qp_b} is a metric on the compact subsets of the metric space (X, qp_b) and thus $(\mathrm{H}(X), h_{qp_b})$ is a quasi-partial b-metric space.

Theorem 0.0.7. *Let (X, qp_b) be a complete quasi-partial b-metric space. Then $(\mathrm{H}(X), h_{qp_b})$ is a complete quasi-partial b-metric space.*

Proof. Assume that (X, qp_b) is a complete quasi-partial b-metric space. To prove $(\mathrm{H}(X), h_{qp_b})$ is a complete quasi-partial b-metric space. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathrm{H}(X), h_{qp_b})$. By the definition of

ISSN: 2581 - 9380
TOJSR. 2025, 4 (1)

the metric h_{qp_b}, we have

$$\begin{aligned}
\lim_{n,m\to\infty} h_{qp_b}(A_n, A_m) &= \lim_{n,m\to\infty} [qp_b(A_n, A_m) \vee qp_b(A_m, A_n)] \\
&= \lim_{n,m\to\infty} qp_b(A_n, A_m) \vee \lim_{n,m\to\infty} qp_b(A_m, A_n) \\
&= \lim_{n,m\to\infty} qp_b(x_n, x_m) \vee \lim_{n,m\to\infty} qp_b(x_m, x_n) \quad \forall x_n \in A_n, x_m \in A_m \text{ and } n, m \in \mathbb{N} \\
&= qp_b(x, x) \vee qp_b(x, x) \quad (\because (X, qp_b) \text{ is complete}) \\
&= qp_b(x, x).
\end{aligned}$$

Similarly, we can have the same result for the case $\lim_{m,n\to\infty} h_{qp_b}(A_m, A_n)$. □

Definition 0.0.13. *Let (X, qp_b) be a quasi-partial b-metric space. Let A and T_n be continuous self-maps that commutes and $T_n(X) \subset A(X)$ for each $n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}$ and satisfy the following condition: for any $x, y \in X$,*

$$\begin{aligned}
qp_b(T_n(x), T_n(y)) \leq\ & \alpha_n qp_b(Ax, Ay) \\
& + \beta qp_b(Ax, T_n(x)) \\
& + \beta_n qp_b(Ay, T_n(y)),
\end{aligned}$$

where $\alpha_n, \beta_n \in (0, 1)$ such that for $s_n \geq 1$ and $s_n(\alpha_n + 2\beta_n) < 1$. Then the system $\{X; A, T_n, n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$ is called an iterated function system consists of finite number of cyclic Ćirić-Reich-Rus contractions with condensation. Here the contractivity factors are $\alpha_n, \beta_n, n = 0, 1, 2, \ldots, N_0$.

Definition 0.0.14. *Let (X, qp_b) be a quasi-partial b-metric space. Let $\{X; A, T_n, n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$ be an iterated function system consists of finite number of cyclic Ćirić-Reich-Rus contractions with condensation. Then the HB operator of the iterated function system is a function $\hat{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by*

$$\hat{T}(B) = \bigcup_{n=0}^{N_0} T_n(B) \forall B \in \mathcal{H}(X)$$

Lemma 0.0.3. *Let (X, qp_b) be a quasi-partial b-metric space and $A, T : X \to X$ be two continuous maps that commutes and $T(X) \subset A(X)$. Also assume that A and T should satisfy the cyclic Ćirić-Reich-Rus contraction condition with contractivity factors α, β. Then the mapping $\hat{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by $\hat{T}(B) = \{T(x) : x \in B\} \forall B \in \mathcal{H}(X)$ is a cyclic Ćirić-Reich-Rus contraction on $(\mathcal{H}(X), h_{qp_b})$ with the same contractivity ratios α, β.*

Proof. Let $B, C \in \mathcal{H}(X)$.

The Oxford Journal of Science and Research

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TOJSR. 2025, 4 (1)

$$\begin{aligned} h_{qp_b}(T(B), T(C)) &= qp_b(T(B), T(C)) \\ &\quad \vee qp_b(T(C), T(B)) \\ &\leq [\alpha qp_b(A(B), A(C)) + \beta qp_b(A(B), T(B)) \\ &\quad + \beta qp_b(A(C), T(C))] \vee \\ &\quad [\alpha qp_b(A(C), A(B)) + \beta qp_b(A(C), T(C)) \\ &\quad + \beta qp_b(A(B), T(B))] \\ &\leq \alpha[qp_b(A(B), A(C)) \vee \\ &\quad qp_b(A(C), A(B))] + \beta[qp_b(A(C), T(B)) \\ &\quad + qp_b(A(C), T(C))] \vee \beta[qp_b(A(C), T(C)) \\ &\quad + qp_b(A(B), T(B))] \\ &= \alpha h_{qp_b}(A(B), A(C)) + \beta qp_b(A(B), T(B)) \\ &\quad + \beta qp_b(A(C), T(C)) \\ &\leq \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), T(B)) \\ &\quad + \beta h_{qp_b}(A(C), T(C)) \end{aligned}$$

i.e., $h_{qp_b}(T(B), T(C)) \leq \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), T(B)) + \beta h_{qp_b}(A(C), T(C))$ □

Main Result

Attractor of the iterated function system constructed using cyclic Ćirić-Reich-Rus contractions with condensation

Lemma 0.0.4. *Let* (X, qp_b) *be a quasi-partial b-metric space. Let* $\{X; A, T_n, n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$ *be an iterated function system consists of finite number of cyclic Ćirić-Reich-Rus contractions with condensation. Then the mapping* $\hat{T} : \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by* $\hat{T}(B) = \bigcup_{n=0}^{N_0} T_n(B) \forall B \in \mathcal{H}(X)$ *is a cyclic Ćirić-Reich-Rus contraction with contractivity factor* $\alpha, \beta = max\{\alpha_n, \beta_n : n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$.

Proof: We show the result by induction on n. Suppose $n = 0$. Then $\hat{T}(B) = T_0(B) \forall B \in \mathcal{H}(X)$. Since condensation is a cyclic Ćirić-Reich-Rus contraction, the result is true for this case. If $n = 1$, the result follows from 0.0.3. If $n = 2$, we see that

TOJSR
The Oxford Journal of Science and Research
ISSN: 2581 - 9380
TOJSR. 2025, 4 (1)

$$
\begin{aligned}
h_{qp_b}(\hat{T}(B), \hat{T}(C)) &= h_{qp_b}(T_1(B) \cup T_2(B), T_1(C) \cup T_2(C)) \\
&\leq h_{qp_b}(T_1(B), T_1(C)) \vee h_{qp_b}(T_2(B), T_2(C)) \\
&= [qp_b(T_1(B), T_1(C)) \vee qp_b(T_1(C), T_1(B))] \vee [qp_b(T_2(B), T_2(C)) \\
&\quad \vee qp_b(T_2(C), T_2(B))] \\
&\leq [(\alpha_1 qp_b(A(B), A(C)) + \beta_1 qp_b(A(B), T_1(B)) + \beta_1 qp_b(A(C), T_1(C))) \\
&\quad \vee(\alpha_1 qp_b(A(C), A(B)) + \beta_1 qp_b(A(C), T_1(C)) + \beta_1 qp_b(A(B), T_1(B)))] \\
&\quad \vee[(\alpha_2 qp_b(A(B), A(C)) + \beta_2 qp_b(A(B), T_1(B)) + \beta_2 qp_b(A(C), T_1(C))) \\
&\quad \vee(\alpha_2 qp_b(A(C), A(B)) + \beta_2 qp_b(A(C), T_1(C)) + \beta_2 qp_b(A(B), T_1(B)))] \\
&= \{\alpha_1[qp_b(A(B), A(C)) \vee qp_b(A(C), A(B))] + [\beta_1 qp_b(A(B), T_1(B)) \\
&\quad +\beta_1 qp_b(A(C), T_1(C)) \vee \beta_1 qp_b(A(C), T_1(C)) + \beta_1 qp_b(A(B), T_1(B))]\} \\
&\quad \{\alpha_2[qp_b(A(B), A(C)) \vee qp_b(A(C), A(B))] + [\beta_2 qp_b(A(B), T_2(B)) \\
&\quad +\beta_2 qp_b(A(C), T_2(C)) \vee \beta_2 qp_b(A(C), T_2(C)) + \beta_2 qp_b(A(B), T_2(B))]\} \\
&= [\alpha_1 h_{qp_b}(A(B), A(C)) + \beta_1 qp_b(A(B), T_1(B)) + \beta_1 qp_b(A(C), T_1(C))] \\
&\quad \vee[\alpha_2 h_{qp_b}(A(B), A(C)) + \beta_2 qp_b(A(B), T_2(B)) + \beta_2 qp_b(A(C), T_2(C))] \\
&\leq [\alpha_1 h_{qp_b}(A(B), A(C)) + \beta_1 h_{qp_b}(A(B), T_1(B)) + \beta_1 h_{qp_b}(A(C), T_1(C))] \\
&\quad \vee[\alpha_2 h_{qp_b}(A(B), A(C)) + \beta_2 h_{qp_b}(A(B), T_2(B)) + \beta_2 qp_b(A(C), T_2(C))] \\
&\leq (\alpha_1 \vee \alpha_2)[h_{qp_b}(A(B), A(C)) \vee h_{qp_b}(A(B), A(C))] + \\
&\quad (\beta_1 \vee \beta_2)[h_{qp_b}(A(B), T_1(B)) \vee h_{qp_b}(A(B), T_2(B))] + \\
&\quad (\beta_1 \vee \beta_2)[h_{qp_b}(A(C), T_1(C)) \vee h_{qp_b}(A(C), T_2(C))] \\
&= \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), T_1(B) \cup T_2(B)) \\
&\quad +\beta h_{qp_b}(A(C), T_1(C) \cup T_2(C)) \\
&= \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), \hat{T}(B)) + \beta h_{qp_b}(A(C), \hat{T}(C))
\end{aligned}
$$

So the result is true for the case $n = 2$. Assume that the lemma is valid for $N_0 - 1$. i.e.,

$$
\begin{aligned}
h_{qp_b}(\hat{T}(B), \hat{T}(C)) &= h_{qp_b}\left(\bigcup_{n=1}^{N_0-1} T_n(B), \bigcup_{n=1}^{N_0-1} T_n(C)\right) \\
&\leq \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}\left(A(B), \bigcup_{n=1}^{N_0-1} T_n(B)\right) + \beta h_{qp_b}\left(A(C), \bigcup_{n=1}^{N_0-1} T_n(C)\right) \\
&= \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), \hat{T}(B)) + \beta h_{qp_b}(A(C), \hat{T}(C))
\end{aligned}
$$

where $\alpha = max\{\alpha_1, \alpha_2, \ldots, \alpha_{N_0-1}\}$, $\beta = \max\{\beta_1, \beta_2, \ldots, \beta_{N_0-1}\}$. Now we prove this for the cases N_0. i.e., $\hat{T}(B) = \bigcup_{n=1}^{N_0-1} T_n(B) \cup T_{N_0}(B)$.

$$
\begin{aligned}
h_{qp_b}(\hat{T}(B), \hat{T}(C)) &= h_{qp_b}\left(\bigcup_{n=1}^{N_0-1} T_n(B) \cup T_{N_0}(B), \bigcup_{n=1}^{N_0-1} T_n(C) \cup T_{N_0}(C)\right) \\
&\leq h_{qp_b}\left(\bigcup_{n=1}^{N_0-1} T_n(B), \bigcup_{n=1}^{N_0-1} T_n(C)\right) \vee h_{qp_b}(T_{N_0}(B), T_{N_0}(C)) \\
&\leq [(\alpha_1 \vee \alpha_2 \vee \ldots \alpha_{N_0-1}) h_{qp_b}(A(B), A(C)) + (\beta_1 \vee \beta_2 \vee \ldots \beta_{N_0-1}) \\
&\quad h_{qp_b}\left(A(B), \bigcup_{n=1}^{N_0-1} T_n(B)\right) + (\beta_1 \vee \beta_2 \vee \ldots \beta_{N_0-1}) h_{qp_b}\left(A(C), \bigcup_{n=1}^{N_0-1} T_n(C)\right)] \\
&\quad \vee [\alpha_{N_0} h_{qp_b}(A(B), A(C)) + \beta_{N_0} h_{qp_b}(A(B), T_{N_0}(B)) + \beta_{N_0} h_{qp_b}(A(C), T_{N_0}(C))] \\
&\leq (\alpha_1 \vee \alpha_2 \vee \ldots \alpha_{N_0-1} \vee \alpha_{N_0}) h_{qp_b}(A(B), A(C)) + (\beta_1 \vee \beta_2 \vee \ldots \beta_{N_0-1} \vee \beta_{N_0}) \\
&\quad \left(h_{qp_b}\left(A(B), \bigcup_{n=1}^{N_0-1} T_n(B)\right) \vee h_{qp_b}(A(B), T_{N_0}(B))\right) + \\
&\quad (\beta_1 \vee \beta_2 \vee \ldots \beta_{N_0-1} \vee \beta_{N_0}) \left(h_{qp_b}\left(A(C), \bigcup_{n=1}^{N_0-1} T_n(C)\right) \vee h_{qp_b}(A(C), T_{N_0}(C))\right) \\
&= \alpha h_{qp_b}(A(B), A(C)) + \beta h_{qp_b}(A(B), \hat{T}(B)) + \beta h_{qp_b}(A(C), \hat{T}(C)).
\end{aligned}
$$

Therefore by mathematical induction the lemma holds for all n.

Theorem 0.0.8. *Let (X, qp_b) be a complete quasi-partial b-metric space. Let $\{X; A, T_n, n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$ be an iterated function system consists of finite number of cyclic Ćirić-Reich-Rus contractions with condensation. Then the mapping $\hat{T} : \mathrm{H}(X) \to \mathrm{H}(X)$ defined by $\hat{T}(B) = \bigcup_{n=0}^{N_0} T_n(B) \forall B \in \mathrm{H}(X)$ is a cyclic Ćirić-Reich-Rus contraction with contractivity factor $\alpha, \beta = max\{\alpha_n, \beta_n : n = 0, 1, 2, \ldots, N_0; N_0 \in \mathbb{N}\}$ on the complete quasi-partial b-metric space $(\mathrm{H}(X), h_{qp_b})$. Also it has a unique fixed point, which is called an attractor, $A_\infty \in \mathrm{H}(X)$ lays down $A_\infty = \hat{T}(A_\infty) = \bigcup_{n=0}^{N_0} T_n(A_\infty)$ and is given by $A_\infty = \lim_{n\to\infty} \hat{T}^{\circ n}(A_\infty)$.*

Proof. Since (X, qp_b) is a complete quasi-partial b-metric space, then by Theorem 0.0.7, $(\mathrm{H}(X), h_{qp_b})$ is a complete Hausdorff quasi-partial b-metric space. Also by Lemma 0.0.4, the HB operator $\hat{T}$ is a cyclic Ćirić-Reich-Rus contraction transformation. Then from Theorem 0.0.1, we can show that $\hat{T}$ has a unique fixed point. □

Definition 0.0.15. *The fixed point $A_\infty \in \mathrm{H}(X)$ of the HB operator $\hat{T}$ described in Theorem 0.0.8 is called the attractor (fractal) of the iterated function system.*

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TOJSR. 2025, 4 (1)

A new methodology of construction of an attractor on the hyperspace of quasi-partial b-metric space

In this section a novel method of constructing an attractor of a self-mapping defined on the hyperspace of quasi-partial b-metric space is derived. Before going to see the anticipated result, we focus on some definitions.

Definition 0.0.16. *[21] Let (X, qp_b) be a quasi-partial b-metric space, and let $\epsilon > 0$ be given, then a set $A \subset X$ is called an ϵ-net of (X, qp_b) if given any x in X there is atleast one point a in A such that $x \in B_{qp_b}(a, \epsilon)$, where $B_{qp_b}(a, \epsilon) = \{y \in X : qp_b(a, y) < \epsilon$ and $qp_b(y, a) < \epsilon\}$.*
If the set A is finite then A is called a finite ϵ-net of (X, qp_b).
If A is an ϵ-net then $X = \bigcup_{a\in A} B_{qp_b(a,\epsilon)}$.

Definition 0.0.17. *[21] A quasi-partial b-metric space (X, qp_b) is called qp_b-totally bounded if for every $\epsilon > 0$ there exists a finite ϵ-net.*

Definition 0.0.18. *[21] A quasi-partial b-metric space (X, qp_b) is said to be a compact quasi-partial b-metric if it is qp_b-complete and qp_b-totally bounded.*

Now we will perceive the superior result, which gives a new way to attain an attractor of a self-mapping on the hyperspace of a quasi-partial b-metric space.

Theorem 0.0.9. *Let (Y, qp_b) be a complete quasi-partial b-metric space. Let $X \subset Y$ be non-empty and compact. Let $f : X \to Y$ be continuous and such that $X \subset f(X)$. Then*

(1) A transformation $F : \mathbb{H}(X) \to \mathbb{H}(X)$ is defined by

$$F(A) = f^{-1}(A) \quad \text{for all } A \subset \mathbb{H}(X)$$

(2) F possesses a fixed point $A \in \mathbb{H}(X)$, given by

$$A = \bigcap_{n=0}^{\infty} f^{\circ(-n)}(X) = \lim_{n\to\infty} F^{\circ n}(X)$$

Where $f^{\circ(-n)} = (f^{\circ n})^{-1} = (\underbrace{f \circ f \circ \cdots f}_{n-times})^{-1}$ and $F^{\circ n} = (\underbrace{F \circ F \circ \cdots F}_{n-times})$.

Proof. We discuss the subdivisions one by one.

(1) We will show that F maps $\mathbb{H}(X)$ into $\mathbb{H}(X)$. Let B be any arbitrary set in $\mathbb{H}(X)$. Since $X \subset f(X)$, $f^{-1}(B) \subset X$ and that $f^{-1}(B)$ is non-empty. Because B is compact, so it is a closed set in the metric space (X, qp_b). It follows that $X\backslash B$ is open. Since f is continuous, $f^{-1}(X\backslash B)$ is open. Since $f(X) \supseteq X \supseteq B$, it follows that $f^{-1}(B) = X\backslash f^{-1}(X\backslash B)$. Hence $f^{-1}(B)$ is closed in the metric space (X, qp_b). Therefore the condition X is compact implies that $f^{-1}(B)$ is compact.

(2) Since $X \subseteq f(X)$, it follows that

$$X \supseteq f^{\circ(-1)}(X) \qquad (1)$$

Applying $f^{\circ(-n)}$ to both sides of the equation (1), we have

$$X \supseteq f^{\circ(-1)}(X) \supseteq f^{\circ(-2)}(X) \supseteq \ldots \supseteq f^{\circ(-n)}(X) \supseteq \ldots$$

It follows that $\{f^{\circ(-n)}(X)\}$ is a Cauchy sequence in $\mathcal{H}(X)$. The compactness of X implies that X is closed and completeness of Y yeilds that (X, qp_b) is complete. Then by Theorem 0.0.7 the Cauchy sequence possesses a limit $A \in \mathcal{H}(X)$, given by

$$A = \bigcap_{n=0}^{\infty} f^{\circ(-n)}(X) = \lim_{n\to\infty} F^{\circ n}(X).$$

It remains to show that A is a fixed point of f. That is to prove that $f(A) = A$.

$$f\left(\bigcap_{n=0}^{\infty} f^{\circ(-n)}(X)\right) = \bigcap_{n=0}^{\infty} f^{\circ(-n)}(X)$$

$$f\left(\bigcap_{n=0}^{\infty} A_n\right) = \bigcap_{n=0}^{\infty} A_n \quad \text{where } A_n = f^{\circ(-n)}(X),\ n = 1, 2, \ldots$$

$$\bigcap_{n=0}^{\infty} A_n = f^{\circ(-1)}\left(\bigcap_{n=0}^{\infty} A_n\right)$$

First we prove that $\bigcap_{n=0}^{\infty} A_n \subseteq f^{\circ(-1)}(\bigcap_{n=0}^{\infty} A_n)$. Suppose that $x \in \bigcap_{n=0}^{\infty} A_n$. Then $x = f^{-1}(A_n)$, $n = 1, 2, \ldots$. It follows that there is $y_n \in A_n$ such that $f(y_n) = x$ for $n = 1, 2, \ldots$. The sequence $\{y_{[n]}\}$ possesses a convergent subsequence. Let the limit of this subsequence be y. Then $y \in A_n, n = 1, 2, \ldots$ and so $y \in \bigcap_{n=0}^{\infty} A_n$. Since f is continuous if follows that $f(y) = x$. Hence $x \in f^{\circ(-1)}\left(\bigcap_{n=0}^{\infty} A_n\right)$.

$$\Rightarrow \bigcap_{n=0}^{\infty} A_n \subseteq f^{\circ(-1)}\left(\bigcap_{n=0}^{\infty} A_n\right).$$

To prove the inclusion the other way around. Suppose $x \in f^{\circ(-1)}\left(\bigcap_{n=0}^{\infty} A_n\right)$. Then there is $y \in \bigcap A_n$ such that $x = f^{\circ(-1)}(y)\,\forall n = 1, 2, \ldots$. It follows that $f^{\circ(-1)}(A_n) = A_{n+1}$ for $n = 1, 2, \ldots$.

$$\Rightarrow x \in \bigcap_{n=0}^{\infty} A_n$$

$$\Rightarrow f^{\circ(-1)}\left(\bigcap_{n=0}^{\infty} A_n\right) \subseteq \bigcap_{n=0}^{\infty} A_n$$

□

Conclusion

We have considered an iterated function system containing cyclic Ćirić-Reich-Rus contraction mappings with condensation. Using the common fixed point theorem for such contraction mappings on quasi-partial b-metric space, we developed the HB theory to construct a novel fractal set. Also, we gave a new result that produces the fractal as a fixed point of the self-mapping defined on the hyperspace $\mathcal{H}(X)$, where X is a complete quasi-partial b-metric space.

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